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# Large-mass behaviour of loop variables in Abelian Maxwell–Chern–Simons theory

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**Abstract.** The large-mass behaviour of loop variables in Maxwell–Chern–Simons theory is analysed by means of a gauge-field transformation which allows us to reset the Maxwell–Chern–Simons action to pure Chern–Simons.

## 1. Introduction

In recent works [1, 2] it has been established that three-dimensional gauge theories in the presence of the topological Chern–Simons term can be cast in the form of a pure Chern–Simons action through a local covariant redefinition of the gauge connection. For instance, in the case of the standard Yang–Mills term  $\int F F$ , we obtain

$$S_{\text{CS}}(A) + \frac{1}{4m} \text{tr} \int d^3x F_{\mu\nu} F^{\mu\nu} = S_{\text{CS}}(\widehat{A}) \quad (1.1)$$

with

$$\widehat{A}_\mu = A_\mu + \sum_{n=1}^{\infty} \frac{1}{m^n} \vartheta_\mu^n \quad (1.2)$$

and

$$S_{\text{CS}}(A) = \frac{1}{2} \text{tr} \int d^3x \varepsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + \frac{2}{3} g A_\mu A_\nu A_\rho). \quad (1.3)$$

The two parameters  $g$ ,  $m$  in the expressions (1.1) and (1.3) can be identified, respectively, with the gauge coupling and with the topological mass. The coefficients  $\vartheta_\mu^n$  in equation (1.2) have been proven [2] to transform covariantly under gauge transformations and can be expressed in terms of the curvature  $F_{\mu\nu}$  and its covariant derivatives. This implies that the redefined field  $\widehat{A}_\mu$  is still a connection. This property has led to an attractive geometrical interpretation of the Chern–Simons term as a kind of topological generator for the classical Yang–Mills-type actions [2].

The existence of the transformation (1.2) has also been exploited at the quantum level. Several results have been obtained for the quantum effective actions of different systems. In the case of the massive topological Yang–Mills (1.1), the redefinition (1.2) has allowed for a purely algebraic proof of the ultraviolet finiteness of the model [2]. Moreover, in the case of the effective action corresponding to the Abelian fermionic determinant of a massive two-component spinor field, equation (1.2) has been extended at the quantum level in order to

account for the nonlocal quantum corrections [3]. As a consequence, it has been possible to prove that the infinite number of one-loop diagrams corresponding to the perturbative expansion of the fermionic determinant can be reabsorbed into the pure Chern–Simons, up to a field redefinition [3].

On the other hand, it is known that pure Chern–Simons theory can be recovered as the infinite-mass limit  $m \rightarrow \infty$  of the massive topological Yang–Mills action (1.1). This property has been proven to hold for both the 1PI effective action [4] and for the vacuum expectation value of the Wilson loop [5]. We thus expect that, for a finite large value of the mass parameter  $m$  (i.e. a low-energy regime), the effects of the presence of the Yang–Mills term show up in the form of a power series in  $1/m$ . Furthermore, being the field redefinition (1.2) a local expansion in  $1/m$ , it gives us a natural way to deal with the large-mass corrections which affect the vacuum expectation value of the observables, i.e. of the nontrivial gauge-invariant quantities.

This is the aim of this paper. In particular, we shall investigate whether the redefinition (1.2) can be used as an effective computational tool in order to characterize the large-mass behaviour of the observables. In this work we shall restrict ourselves to the Abelian case. The main idea is to use equation (1.2) as a change of variables in the path integral. In doing this, one picks up the Jacobian of the transformation and one has to re-express the observable under consideration in terms of the redefined field  $\hat{A}_\mu$ , which, now being a power series in  $1/m$ , will systematically produce a local expansion of the observable in powers of  $1/m$ . However, such a change of variables in the path integral leads the Boltzmann weight to take the form of the pure Chern–Simons action. This procedure may therefore have the practical advantage of performing computations by making use of the Chern–Simons propagator

$$\mathcal{G}_{\mu\nu}^{\text{CS}} = -\frac{1}{4\pi} \varepsilon_{\mu\nu\rho} \partial^\rho \frac{1}{|x-y|} = \frac{1}{4\pi} \varepsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (1.4)$$

instead of the more complicated one corresponding to the quadratic part of the action (1.1), i.e.

$$\mathcal{G}_{\mu\nu}^{\text{MCS}} = \frac{1}{4\pi} \varepsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} + \frac{m}{4\pi} \frac{e^{-m|x-y|}}{|x-y|} \left( g_{\mu\nu} - \frac{1}{m} \varepsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^2} (1+m|x-y|) \right). \quad (1.5)$$

In other words, in the case of a large value of  $m$ , the field redefinition (1.2) will allow us to shift the mass dependence from the Boltzmann weight directly to the observable, so that the expectation value can be obtained by making use of the pure Chern–Simons propagator. As an example of this set-up we shall use the field redefinition (1.2) in the case of the Abelian Maxwell–Chern–Simons theory in flat spacetime

$$\mathcal{S}_{\text{MCS}}(A) = \frac{1}{2} \int d^3x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{1}{4m} \int d^3x F_{\mu\nu} F^{\mu\nu} \quad (1.6)$$

in order to compute the large-mass corrections to the loop factor [6, 7]

$$\mathcal{W}(\gamma, \gamma') = e^{-\oint_\gamma dx^\mu \oint_{\gamma'} dy^\nu \langle A_\mu(x) A_\nu(y) \rangle} \quad (1.7)$$

where  $\gamma, \gamma'$  are two distinct (nonintersecting) smooth closed oriented curves (which define a two-component link  $L(\gamma, \gamma')$  [8]). As discussed by [7], the relevance of the factor  $\mathcal{W}(\gamma, \gamma')$  is due to the fact that it carries the information concerning the statistics of the quantum fluctuations of the  $(2+1)$ -dimensional Abelian Higgs model, thanks to a random-walk representation for the gauge-invariant Green's functions. In particular, by making use of the redefinition (1.2), we shall be able to prove that all the expected corrections in the inverse of the mass parameter  $m$  are actually absent, provided  $\gamma, \gamma'$  are two disjoint curves, one of which may wind around the other. In other words, the double line integral  $\oint_\gamma dx^\mu \oint_{\gamma'} dy^\nu \langle A_\mu(x) A_\nu(y) \rangle$  computed with the Maxwell–Chern–Simons action yields, in the large-mass limit, the linking number  $\chi(\gamma, \gamma')$ .

It is worth remarking here that this set-up could have useful applications for the three-dimensional bosonization of fermionic systems for a large value of the fermion mass. We recall in fact that the large-mass expansion of the so-called three-dimensional fermionic determinant is one of the basic ingredients of our present understanding of three-dimensional bosonization [9].

The paper is organized as follows. In section 2 we discuss the main properties of the field redefinition (1.2) in the case of the Abelian Maxwell–Chern–Simons theory. In section 3 the details of the computation of the double line integral  $\oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \langle A_{\mu}(x) A_{\nu}(y) \rangle$  will be given. In section 4 we discuss the extension of this result to a more general class of Abelian actions as well as to a generic link  $L(\gamma_1, \dots, \gamma_n)$ . Section 5 is devoted to the case in which  $\gamma$  and  $\gamma'$  identify the same curve, expression (1.7) becoming there the expectation value of the Abelian Wilson loop [5, 6, 10, 11]. The example of a planar loop will be reported in detail.

Although the present work deals with the large-mass behaviour of the loop variables, a simple framework for the case of the small-mass corrections is provided in the appendix.

## 2. The Maxwell–Chern–Simons action

As already mentioned, the Abelian Maxwell–Chern–Simons action can be reset to a pure Chern–Simons term

$$S_{\text{MCS}}(A) = \frac{1}{2} \int d^3x \varepsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} A_{\rho} + \frac{1}{4m} \int d^3x F_{\mu\nu} F^{\mu\nu} = S_{\text{CS}}(\widehat{A}) \quad (2.1)$$

through a suitable gauge field redefinition of the kind

$$\widehat{A}_{\mu} = A_{\mu} + \sum_{n=1}^{\infty} \frac{1}{m^n} \vartheta_{\mu}^n. \quad (2.2)$$

As proven in [2] by using BRST cohomological techniques, the coefficients  $\vartheta_{\mu}^n$  turn out to depend only on the field strength and its derivatives. In the present Abelian case their computation is rather straightforward. For instance, the first six coefficients are found to be

$$\begin{aligned} \vartheta_{\mu}^1 &= \frac{1}{4} \varepsilon_{\mu\nu\rho} F^{\nu\rho} & \vartheta_{\mu}^2 &= -\frac{1}{8} \partial^{\nu} F_{\mu\nu} \\ \vartheta_{\mu}^3 &= -\frac{1}{32} \varepsilon_{\mu\nu\rho} \partial^2 F^{\nu\rho} & \vartheta_{\mu}^4 &= \frac{5}{128} \partial^2 \partial^{\nu} F_{\mu\nu} \\ \vartheta_{\mu}^5 &= \frac{7}{512} \varepsilon_{\mu\nu\rho} \partial^4 F^{\nu\rho} & \vartheta_{\mu}^6 &= -\frac{21}{1024} \partial^4 \partial^{\nu} F_{\mu\nu}. \end{aligned} \quad (2.3)$$

Although the higher-order coefficients can be easily obtained, the above expressions give us a simple and clear understanding of the general properties of the  $\vartheta_{\mu}^n$ 's. They can be summarized as follows:

- the coefficients  $\vartheta_{\mu}^n$  are divergenceless, i.e.

$$\partial^{\mu} \vartheta_{\mu}^n = 0 \quad (2.4)$$

- they are gauge invariant and depend linearly on the gauge field  $A_{\mu}$ .

As one can easily understand, these properties follow from the Abelian character of the Maxwell–Chern–Simons action (2.1). They will turn out to be of great relevance in order to compute the large-mass behaviour of  $\mathcal{W}(\gamma, \gamma')$ . In particular, from equation (2.4) it follows that the general form of the field transformation (2.2) can be written in terms of the two transverse projectors  $\varepsilon_{\mu\nu\rho} \partial^{\rho}$  and  $(g_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu})$  as

$$\widehat{A}_{\mu} = A_{\mu} + \frac{1}{m} \varepsilon_{\mu\nu\rho} f(\partial^2/m^2) \partial^{\nu} A^{\rho} - \frac{1}{m^2} h(\partial^2/m^2) (g_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu}) A^{\nu} \quad (2.5)$$

where  $f$  and  $h$  are power series in the three-dimensional Laplacian

$$\begin{aligned}
 f(\partial^2/m^2) &= \frac{1}{2} - \frac{1}{16} \frac{\partial^2}{m^2} + \frac{7}{256} \frac{\partial^4}{m^4} + \dots \\
 h(\partial^2/m^2) &= -\frac{1}{8} + \frac{5}{128} \frac{\partial^2}{m^2} - \frac{21}{1024} \frac{\partial^4}{m^4} + \dots
 \end{aligned}
 \tag{2.6}$$

Observe that from equation (2.4) it follows that the two gauge connections  $\widehat{A}_\mu$  and  $A_\mu$  have the same divergence,

$$\partial^\mu \widehat{A}_\mu = \partial^\mu A_\mu
 \tag{2.7}$$

which implies that, in a covariant Lorentz-type gauge condition, the gauge-fixing term remains unchanged when one moves from  $A_\mu$  to  $\widehat{A}_\mu$ .

Let us give here, for later use, the coefficients of the inverse transformation (2.3) which has, of course, the same general form of equation (2.5):

$$A_\mu = \widehat{A}_\mu + \frac{1}{m} \varepsilon_{\mu\nu\rho} \widehat{f}(\partial^2/m^2) \partial^\nu \widehat{A}^\rho - \frac{1}{m^2} \widehat{h}(\partial^2/m^2) (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \widehat{A}^\nu
 \tag{2.8}$$

with

$$\begin{aligned}
 \widehat{f}(\partial^2/m^2) &= -\frac{1}{2} + \frac{5}{16} \frac{\partial^2}{m^2} - \frac{63}{256} \frac{\partial^4}{m^4} + \dots \\
 \widehat{h}(\partial^2/m^2) &= \frac{3}{8} - \frac{35}{128} \frac{\partial^2}{m^2} + \frac{231}{1024} \frac{\partial^4}{m^4} + \dots
 \end{aligned}
 \tag{2.9}$$

It should also be remarked that equations (2.5) and (2.8), being linear in the fields  $A_\mu, \widehat{A}_\mu$ , imply that the Jacobian  $\det(\delta A_\nu / \delta \widehat{A}_\mu)$  corresponding to the change of variables  $A_\mu \rightarrow \widehat{A}_\mu$  is field independent, and therefore it does not contribute to the transformed measure  $\mathcal{D}\widehat{A}$  in the path integral.

We are now ready to evaluate the large-mass effects to the double line integral of the expression (1.7). This will be the task of the next section.

### 3. Computation of the double-line integral

In order to compute the line integral  $\oint_{\gamma'} dx^\mu \oint_{\gamma} dy^\nu \langle A_\mu(x) A_\nu(y) \rangle$  in the Maxwell–Chern–Simons theory we have first to fix the gauge. Adopting a transverse Landau gauge, we can write

$$\oint_{\gamma} \oint_{\gamma'} \langle A(x) A(y) \rangle_{\text{MCS}} = \frac{\int \mathcal{D}A \mathcal{D}b \oint_{\gamma} dx^\mu \oint_{\gamma'} dy^\nu A_\mu(x) A_\nu(y) e^{i(S_{\text{MCS}}(A) + \int d^3x b \partial A)}}{\int \mathcal{D}A \mathcal{D}b e^{i(S_{\text{MCS}}(A) + \int d^3x b \partial A)}}
 \tag{3.1}$$

where  $b$  is the Lagrange multiplier implementing the gauge condition. Let us now perform the change of variables (2.8). Moreover, recalling that the corresponding Jacobian is field independent and that, from equation (2.7), the Landau gauge condition is left invariant, we obtain

$$\oint_{\gamma} dx^\mu \oint_{\gamma'} dy^\nu \langle A_\mu(x) A_\nu(y) \rangle_{\text{MCS}} = \oint_{\gamma} dx^\mu \oint_{\gamma'} dy^\nu \langle A_\mu(\widehat{A}(x)) A_\nu(\widehat{A}(y)) \rangle_{\text{CS}}
 \tag{3.2}$$

where

$$\begin{aligned}
 &\oint_{\gamma} dx^\mu \oint_{\gamma'} dy^\nu \langle A_\mu(\widehat{A}(x)) A_\nu(\widehat{A}(y)) \rangle_{\text{CS}} \\
 &= \frac{\int \mathcal{D}\widehat{A} \mathcal{D}b \oint_{\gamma} dx^\mu \oint_{\gamma'} dy^\nu (\widehat{A}_\mu(x)) A_\nu(\widehat{A}(y)) e^{i(S_{\text{CS}}(\widehat{A}) + \int d^3x b \partial \widehat{A})}}{\int \mathcal{D}\widehat{A} \mathcal{D}b e^{i(S_{\text{CS}}(\widehat{A}) + \int d^3x b \partial \widehat{A})}}.
 \end{aligned}
 \tag{3.3}$$

We see therefore that the expectation value of  $\oint_{\gamma} \oint_{\gamma'} \langle A(x) A(y) \rangle_{\text{MCS}}$  in the Maxwell–Chern–Simons theory can be obtained by computing the expectation value of the transformed quantity  $\oint_{\gamma} \oint_{\gamma'} \langle A(\widehat{A}(x)) A(\widehat{A}(y)) \rangle_{\text{CS}}$  in the (topological) pure Chern–Simons theory. Therefore

$$\oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \langle A_{\mu}(\widehat{A}(x)) A_{\nu}(\widehat{A}(y)) \rangle_{\text{CS}} = \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \Omega_{\mu\sigma}(x) \Omega_{\nu\lambda}(y) \langle \widehat{A}^{\sigma}(x) \widehat{A}^{\lambda}(y) \rangle_{\text{CS}} \tag{3.4}$$

with  $\Omega_{\mu\sigma}(x)$  given by

$$\Omega_{\mu\sigma}(x) = \left( g_{\mu\sigma} + \frac{1}{m} \varepsilon_{\mu\rho\sigma} \widehat{f}(\partial_x^2/m^2) \partial_x^{\rho} - \frac{1}{m^2} \widehat{h}(\partial_x^2/m^2) (g_{\mu\sigma} \partial^2 - \partial_{\mu} \partial_{\sigma})_x \right). \tag{3.5}$$

In order to evaluate expression (3.4) let us recall that the Landau propagator of the pure Chern–Simons theory,

$$\langle \widehat{A}^{\sigma}(x) \widehat{A}^{\lambda}(y) \rangle_{\text{CS}} = -\frac{1}{4\pi} \varepsilon^{\sigma\lambda\tau} \partial_{\tau} \frac{1}{|x-y|} \tag{3.6}$$

is transverse,

$$\partial_{\sigma} \langle \widehat{A}^{\sigma}(x) \widehat{A}^{\lambda}(y) \rangle_{\text{CS}} = 0 \tag{3.7}$$

and that, from

$$\partial^2 \frac{1}{|x-y|} = -4\pi \delta^3(x-y) \tag{3.8}$$

we obtain

$$\partial^2 \langle \widehat{A}^{\sigma}(x) \widehat{A}^{\lambda}(y) \rangle_{\text{CS}} = 0 \quad \text{for } x \neq y. \tag{3.9}$$

This last identity can be applied directly to equation (3.4), as  $\gamma$  and  $\gamma'$  are two disjoint (nonintersecting) curves. Therefore the points  $x$  and  $y$  will never coincide. As a consequence, all the Laplacians in the factors  $\Omega$  of equation (3.4) can be eliminated. The same occurs for the terms containing the derivatives  $\partial_{\mu} \partial_{\sigma}$  and  $\partial_{\nu} \partial_{\lambda}$ , due to the transversality of the Chern–Simons propagator. Expression (3.4) thus reduces to

$$\begin{aligned} \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \langle A_{\mu}(\widehat{A}(x)) A_{\nu}(\widehat{A}(y)) \rangle_{\text{CS}} &= -\frac{1}{4\pi} \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \varepsilon_{\mu\nu\tau} \partial^{\tau} \frac{1}{|x-y|} \\ &\quad - \frac{1}{4\pi m} \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \varepsilon_{\mu\rho\sigma} \varepsilon^{\sigma\lambda}_{\nu} \partial^{\rho} \partial_{\lambda} \frac{1}{|x-y|} \\ &\quad + \frac{1}{16\pi m^2} \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \varepsilon_{\mu\rho\sigma} \varepsilon_{\nu\tau\lambda} \varepsilon^{\sigma\lambda\alpha} \partial^{\rho} \partial^{\tau} \partial_{\alpha} \frac{1}{|x-y|} \\ &= -\frac{1}{4\pi} \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \varepsilon_{\mu\nu\tau} \partial^{\tau} \frac{1}{|x-y|} \end{aligned} \tag{3.10}$$

where all the  $1/m$ -dependent terms turn out to identically vanish or to yield a total derivative upon contraction of the  $\varepsilon_{\mu\nu\rho}$  factors. For the final result we therefore obtain [8]

$$\oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \langle A_{\mu}(x) A_{\nu}(y) \rangle_{\text{MCS}} = \chi(\gamma, \gamma') \tag{3.11}$$

where  $\chi(\gamma, \gamma')$  is the linking number of the two curves  $\gamma$  and  $\gamma'$ . We may see, then, that, as announced, the factor  $\mathcal{W}(\gamma, \gamma')$  is not affected by large-mass corrections in  $1/m$ . As one can easily understand, this is due to the fact that the two curves do not intersect each other. It should also be remarked that the use of the transformation (2.5) has allowed us to perform the computations straightforwardly by making use of the properties (3.7) and (3.9) of the pure Chern–Simons propagator.

#### 4. Generalization

Following the algebraic cohomological set-up of [1, 2], it follows that the above result (3.11) can be easily extended to the case in which we add to the Maxwell–Chern–Simons action (2.1) higher derivatives terms of the type

$$\frac{\alpha_n}{m^{2n+1}} \int d^3x F_{\mu\nu} (\partial^2)^n F^{\mu\nu} \quad \frac{\beta_n}{m^{2n}} \int d^3x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu (\partial^2)^n A_\rho \quad n \geq 1 \quad (4.1)$$

where  $\alpha_n$  and  $\beta_n$  are arbitrary dimensionless parameters.

These terms, being quadratic in the gauge field  $A_\mu$ , can be reabsorbed into the pure Chern–Simons action through a linear field redefinition of the kind (2.5). Everything works as before, with the result that the double line integral  $\oint_\gamma dx^\mu \oint_{\gamma'} dy^\nu \langle A_\mu(x) A_\nu(y) \rangle$  is not affected, in the large-mass limit, by corrections in  $1/m$ , meaning that it is in fact independent of the parameters  $\alpha_n, \beta_n$ . We recall here that the terms of equation (4.1), together with the Maxwell–Chern–Simons action (1.6), appear in the large-mass expansion of the two-point Green's function (i.e. of the spinor vacuum polarization) of the effective action corresponding to the Abelian fermionic determinant of a two-component massive spinor [12].

Finally, it is worth underlining that all the results established here can be generalized straightforwardly to a generic line integral  $I(\gamma_1, \dots, \gamma_n)$  of the type

$$I(\gamma_1, \dots, \gamma_n) = \oint_{\gamma_1} dx_1^{\mu_1} \oint_{\gamma_2} dx_2^{\mu_2} \dots \oint_{\gamma_n} dx_n^{\mu_n} \langle A_{\mu_1}(x_1) \dots A_{\mu_n}(x_n) \rangle \quad (4.2)$$

where the curves  $\gamma_1, \dots, \gamma_n$  belong to a  $n$ -component link  $L(\gamma_1, \dots, \gamma_n)$ .

#### 5. Large-mass behaviour of the Wilson Loop

In this section we consider the degenerate case of the Wilson loop, which amounts to computing, within the Maxwell–Chern–Simons context, the link variable  $I(\gamma, \gamma')$  when  $\gamma$  and  $\gamma'$  both refer to the same curve, that is,

$$I(\gamma, \gamma) = \oint_\gamma dx^\mu \oint_\gamma dy^\nu \langle A_\mu(x) A_\nu(y) \rangle_{\text{MCS}}. \quad (5.1)$$

It is worth noting that the double line integral (5.1) computed in pure Chern–Simons is finite and can be defined as the writhing number of the curve  $\gamma$  [8, 13]. Moreover, the authors of [5] have been able to show that (5.1) computed in Maxwell–Chern–Simons theory yields, in the infinite mass limit  $m \rightarrow \infty$ , the so-called self-linking number [8, 13, 14]. They have also proven that, by means of a finite renormalization, the self-linking can be converted into the writhing number, thus recovering the previous infinite-mass limit results of [6, 10, 11].

However, to our knowledge, the mass dependence of the Abelian Wilson loop for a finite large value of the mass parameter  $m$  has not yet been completely worked out. Our purpose here is to show how the present set-up can be useful in evaluating the large-mass contributions which affect the expression (5.1).

For large  $m$ , we perform once more the field redefinition (2.8), which allows us to use the Chern–Simons propagator. It leads to an expansion of  $I(\gamma, \gamma)$  in  $1/m$  similar to the one given in (3.10); however, we now cannot eliminate all  $1/m$ -dependent terms, because in the present case the integration variables  $x$  and  $y$  both refer to points along the same curve. The first few

terms in the expansion are seen to be

$$\begin{aligned}
 I(\gamma, \gamma) = & -\frac{1}{4\pi} \oint_{\gamma} dx^{\mu} \oint_{\gamma} dy^{\nu} \varepsilon_{\mu\nu\alpha} \partial_x^{\alpha} \frac{1}{|x-y|} - \frac{1}{m} \oint_{\gamma} dx^{\mu} \oint_{\gamma} dy_{\mu} \delta^3(x-y) \\
 & - \frac{1}{m^2} \oint_{\gamma} dx^{\mu} \oint_{\gamma} dy^{\nu} \varepsilon_{\mu\nu\alpha} \partial_x^{\alpha} \delta^3(x-y) + \frac{1}{m^3} \oint_{\gamma} dx^{\mu} \oint_{\gamma} dy_{\mu} \partial_x^2 \delta^3(x-y) \\
 & + \mathcal{O}\left(\frac{1}{m^4}\right).
 \end{aligned}
 \tag{5.2}$$

The first term defines what is called the writhing number  $w(\gamma)$  of a curve  $\gamma$  [8, 13]. It can be connected to the so-called self-linking number  $L(\gamma)$  by

$$w(\gamma) = L(\gamma) - T(\gamma)$$

where  $T(\gamma)$  is the twist of the framing bundle used to define  $L(\gamma)$  [8, 13]. In the following, we shall use a technique to analyse planar curves, in which case the writhing number vanishes<sup>†</sup>.

In order to calculate the mass-dependent corrections, we first establish a regularization for the delta function through the well known representation

$$\delta^3(x-y) = \lim_{\alpha \rightarrow 0} \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{(p^2)^{\alpha}} e^{ip(x-y)} \tag{5.3}$$

and the  $\alpha \rightarrow 0$  limit will be taken at the end of the computation. For the first contribution of order  $1/m$  in the equation (5.2) we thus write

$$\begin{aligned}
 \mathcal{J}_{\alpha} & \equiv \oint_{\gamma} dx^{\mu} \oint_{\gamma} dy_{\mu} \delta^3(x-y) \\
 & = \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{(p^2)^{\alpha}} f_{\gamma}^{\mu}(p) f_{\mu\gamma}^{*}(p)
 \end{aligned}
 \tag{5.4}$$

where

$$f_{\gamma}^{\mu}(p) = \oint_{\gamma} dx^{\mu} e^{ipx} \tag{5.5}$$

is the Fourier transform of the line element. Observe that for closed curves

$$p_{\mu} f_{\gamma}^{\mu}(p) = 0. \tag{5.6}$$

In order to give a more concrete idea of the evaluation of the integral  $\mathcal{J}_{\alpha}$  we specify the curve defining the loop. Therefore, we shall concentrate on the case in which the curve  $\gamma$  is a circle of radius  $R$ .

Since the curve is planar, we may decompose the momentum variable as in [15, 16]:

$$p_{\mu} = \widehat{p}_{\mu} + p_{\mu}^{\perp}$$

where  $\widehat{p}_{\mu}$  is the projection of  $p_{\mu}$  over the plane containing  $\gamma$ , and  $p_{\mu}^{\perp}$  is the orthogonal component to that plane. From the definition (5.5), we also have that  $f_{\gamma}^{\mu}(p) = f_{\gamma}^{\mu}(\widehat{p})$  and

$$\varepsilon_{\mu\nu\rho} \widehat{p}^{\mu} f_{\gamma}^{\nu}(\widehat{p}) f_{\gamma}^{*\rho}(\widehat{p}) = 0. \tag{5.7}$$

Thus,

$$\mathcal{J}_{\alpha} = \frac{2}{(2\pi)^3} \int d^2 \widehat{p} \left( \int_0^{\infty} dp^{\perp} \frac{1}{((p^{\perp})^2 + \widehat{p}^2)^{\alpha}} \right) |f_{\gamma}(\widehat{p})|^2.$$

The integral in the orthogonal component is evaluated [15, 16] as

$$\int_0^{\infty} dp^{\perp} \frac{1}{((p^{\perp})^2 + \widehat{p}^2)^{\alpha}} = \frac{1}{2} (\widehat{p}^2)^{\frac{1}{2}(1-2\alpha)} \frac{\Gamma(\alpha - \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\alpha)}$$

<sup>†</sup> We recall here that  $\gamma$  is a smooth closed curve without self-intersecting points.



also, for a circle of radius  $R$  [15, 16],

$$|f_\gamma(\widehat{p})|^2 = 4\pi^2 R^2 J_1^2(\widehat{p}R)$$

where  $J_1$  is the Bessel function. By performing the angular integration in  $d^2 p$ , it follows that

$$\mathcal{J}_\alpha = \frac{\Gamma(\alpha - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha)} R^2 \int_0^\infty d\widehat{p} \widehat{p}^{2-2\alpha} J_1^2(\widehat{p}R) \tag{5.8}$$

where now  $\widehat{p}$  denotes the radial variable. The solution to the above integral may be taken from the table [17], leading to the expression

$$\mathcal{J}_\alpha = \frac{R^{2\alpha-1}}{2^{2\alpha-2}} \frac{\Gamma(2\alpha - 2)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(5 - 2\alpha))}{\Gamma(\alpha)\Gamma(\frac{1}{2}(2\alpha + 1))\Gamma(\frac{1}{2}(2\alpha - 1))}. \tag{5.9}$$

The  $\alpha \rightarrow 0$  limit may now be performed, giving finally

$$\mathcal{J}_0 = -\frac{3}{4R}. \tag{5.10}$$

For planar curves, one can show that all even powers in  $1/m$  vanish automatically by making use of equation (5.7). Therefore, the next nonvanishing contribution for  $I(\gamma, \gamma)$  in equation (5.2) is that of order  $1/m^3$ , i.e.

$$\oint_\gamma dx^\mu \oint_\gamma dy^\nu \partial_x^2 \delta^3(x - y) = -\mathcal{J}_{\alpha-1} \tag{5.11}$$

which, using equation (5.9), is computed to be

$$\mathcal{J}_{\alpha-1} = \frac{R^{2\alpha-3}}{2^{2\alpha-4}} \frac{\Gamma(2\alpha - 4)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(7 - 2\alpha))}{\Gamma(\alpha - 1)\Gamma(\frac{1}{2}(2\alpha - 1))\Gamma(\frac{1}{2}(2\alpha - 3))}.$$

Thus, in the limit  $\alpha \rightarrow 0$ , we obtain

$$\mathcal{J}_{-1} = \frac{15}{32R^3}. \tag{5.12}$$

Finally, substituting equations (5.10) and (5.12) into equation (5.2), and remembering that the writhing of a circle is zero, for the large-mass corrections to  $I(\gamma, \gamma)$  in the case of a circle we obtain the expansion

$$I(\gamma, \gamma) = \frac{3}{4mR} - \frac{15}{32m^3R^3} + \mathcal{O}\left(\frac{1}{m^5}\right). \tag{5.13}$$

The higher-order corrections can be evaluated in a similar way and lead to the general formula

$$I(\gamma, \gamma) = \sum_{n=0}^\infty \frac{1}{(mR)^{2n+1}} \frac{1}{n+1} \frac{\Gamma(\frac{1}{2}(5+2n))}{\Gamma(\frac{1}{2}(1-2n))}. \tag{5.14}$$

### 6. Conclusion

We have proven that, in the large-mass limit, the loop factor  $\mathcal{W}(\gamma, \gamma')$  evaluated in the Maxwell–Chern–Simons theory is not affected by  $1/m$ -corrections, provided the two curves  $\gamma, \gamma'$  do not touch each other.

It is worth underlining that this result has been achieved by means of the field redefinition (2.5), which turns out to provide a very useful computational tool in order to deal with the large-mass dependence of loop variables in three-dimensional gauge theories, including the case of the Wilson loop.

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**Appendix. Small-mass expansion**

In this appendix we discuss briefly, along the lines developed in the present paper, the complementary question of the small-mass behaviour of the link variable

$$I(\gamma, \gamma') = \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \langle A_{\mu}(x) A_{\nu}(y) \rangle_{\text{MCS}} \tag{A.1}$$

in the context of the Maxwell–Chern–Simons theory.

For this purpose, we make use of another kind of field redefinition which now allows us to reabsorb the Chern–Simons term into the Maxwell one in the action, that is,

$$\frac{1}{2} \int d^3x \varepsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} A_{\rho} + \frac{1}{4m} \int d^3x F_{\mu\nu} F^{\mu\nu} = \frac{1}{4m} \int d^3x \widehat{F}_{\mu\nu} \widehat{F}^{\mu\nu} \tag{A.2}$$

through a suitable gauge-field transformation of the type

$$A_{\mu} = \widehat{A}_{\mu} + \sum_{n=1}^{\infty} m^n \widehat{\vartheta}_{\mu}^n \tag{A.3}$$

in which the first few coefficients  $\widehat{\vartheta}_{\mu}^n$  are computed to be

$$\begin{aligned} \widehat{\vartheta}_{\mu}^1 &= \frac{1}{4} \varepsilon_{\mu\nu\rho} \frac{1}{\partial^2} F^{\nu\rho} & \widehat{\vartheta}_{\mu}^2 &= -\frac{3}{8} \frac{1}{\partial^4} \partial^{\nu} F_{\mu\nu} \\ \widehat{\vartheta}_{\mu}^3 &= \frac{5}{32} \varepsilon_{\mu\nu\rho} \frac{1}{\partial^4} F^{\nu\rho} & \widehat{\vartheta}_{\mu}^4 &= \frac{13}{64} \frac{1}{\partial^6} \partial^{\nu} F_{\mu\nu}. \end{aligned} \tag{A.4}$$

We observe that, like in the redefinition of section 2, the  $\widehat{\vartheta}_{\mu}^n$ 's are gauge invariant and transverse.

However they are nonlocal, as may be inferred from the presence of the inverse of the Laplacian. This feature will spoil the integral  $I(\gamma, \gamma')$  in equation (A.1) of any topological meaning. As one can easily understand, this is due to the fact that the small-mass behaviour is dominated by the pure Maxwell term which, of course, is not of a topological nature.

Such a change of variables leads to a computation of the link variable within the pure Maxwell theory:

$$I(\gamma, \gamma') = \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \langle A_{\mu}(\widehat{A}(x)) A_{\nu}(\widehat{A}(y)) \rangle_{\text{Maxwell}}.$$

We obtain, therefore, a small-mass expansion for  $I(\gamma, \gamma')$ , whose first contributions are

$$\begin{aligned} I(\gamma, \gamma') &= \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \langle \widehat{A}_{\mu}(x) \widehat{A}_{\nu}(y) \rangle_{\text{Maxwell}} \\ &\quad - \frac{m}{4\pi} \int d^3z \oint_{\gamma} dx^{\mu} \oint_{\gamma'} dy^{\nu} \varepsilon_{\nu\alpha\beta} \frac{1}{|y-z|} \partial_z^{\alpha} \langle \widehat{A}_{\mu}(x) \widehat{A}^{\beta}(z) \rangle_{\text{Maxwell}} + \mathcal{O}(m^2). \end{aligned} \tag{A.5}$$

We may now substitute in the above expression for the Maxwell propagator,

$$\langle \widehat{A}_\mu(x) \widehat{A}_\nu(y) \rangle_{\text{Maxwell}} = \frac{m}{4\pi} \frac{1}{|x-y|} g_{\mu\nu} \quad (\text{A.6})$$

with the result

$$I(\gamma, \gamma') = \frac{m}{4\pi} \oint_\gamma dx^\mu \oint_{\gamma'} dy^\nu \frac{1}{|x-y|} g_{\mu\nu} + \left(\frac{m}{4\pi}\right)^2 \int d^3z \oint_\gamma dx^\mu \oint_{\gamma'} dy^\nu \varepsilon_{\mu\nu\alpha} \frac{1}{|y-z|} \partial_x^\alpha \frac{1}{|x-z|} + \mathcal{O}(m^3). \quad (\text{A.7})$$

Two remarks are in order. First, we see that, as already underlined, the topological character of  $I(\gamma, \gamma')$  is lost in the small-mass regime. Second, the contributions may be evaluated only after specifying the curves  $\gamma$  and  $\gamma'$ .

Let us conclude by underlining that equation (A.7) also applies to the case of the Wilson loop (5.1). For instance, for the first contribution of the expansion (A.7), we obtain

$$I(\gamma, \gamma) = \frac{m}{4\pi} \oint_\gamma dx^\mu \oint_\gamma dy^\nu \frac{1}{|x-y|} + \mathcal{O}(m^2). \quad (\text{A.8})$$

In the case of the circle, the above integral can be evaluated by following the same procedure of section 5, yielding

$$I(\gamma, \gamma) = m \mathcal{J}_{\alpha+1} + \mathcal{O}(m^2) = m \frac{R^{2\alpha+1}}{2^{2\alpha}} \frac{\Gamma(2\alpha) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}(3-2\alpha))}{\Gamma(1+\alpha) \Gamma(\frac{1}{2}(2\alpha+3)) \Gamma(\frac{1}{2}(2\alpha+1))} + \mathcal{O}(m^2). \quad (\text{A.9})$$

As expected, the limit  $\alpha \rightarrow 0$  is singular, due to the presence of  $\Gamma(2\alpha)$ . We have thus recovered the well known divergent contribution to the Wilson loop of the pure Maxwell term in three dimensions [5, 15, 16].

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